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AFFINE SYMPLECTIC GEOMETRY I: APPLICATIONS TO GEOMETRIC INEQUALITIES

BY

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This article is dedicated to my dear friend Julia Rashba

ABSTRACT

We use the integral geometric formulas in the symplectic space of geodesics of a Riemannian manifold to derive various inequalities of isoperimetric type. We give a sharp lower bound for the area of the minimal bubble spanning a spherical curve in $\mathbb{R}^3$. We also present an "inverse Croke inequality" relating the area of the boundary of a complex domain in a Riemannian manifold to the injectivity radius and the volume of the domain. We prove a sharp lower bound for the ground state of the harmonic oscillator operator in $L^2(M)$, where M is a Hadamard manifold.

0.

By affine symplectic geometry we mean the circle of ideas concentrated around the two following principles:

(1) The space of extremals of a given variational problem carries the natural symplectic structure;

(2) The trajectories of a Hamiltonian system, lying on the fixed energy level surface, are determined by this surface rather than by the whole Hamiltonian.

To a certain extent, we are able to replace some "metric" considerations by the "affine" ones. In this paper we study the geometric aspects of the subject. That is, we apply the natural symplectic structure in the space of geodesics of a given Riemannian metric wherever this space is a well-defined manifold to obtain

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various geometric inequalities. This approach was already successfully applied by C. Croke [Cr1], [Cr2] and M. Berger [Ber2], [Be]. In view of further developments and generalizations we reproduce some basic integral geometric formulas [San] from our point of view. The author wants to acknowledge the influence of the beautiful books [Be], [G–S], [B], [San] on the philosophy involved. He is also grateful to Professor Yu. Burago, who pointed him in the direction of Croke's work [Cr1] where the proof of the 4-dimensional isoperimetrical inequality can be found similar to the author's own [Re1]. The results discussed here are the developments of the study started in [Re1], [Re2]. They were reported at the Burago–Zalgaller geometric seminar (LOMI, Leningrad) and the Lindenstrauss–Milman GAFA Israeli seminar, Tel Aviv. The author thanks the participants for discussion and G. Perelman for help with the proof of the lemma in Theorem 11. He is also very grateful to the referee for valuable remarks.

LIST OF RESULTS.

We present a higher dimensional Riemannian analogue of the Pohl formula (Proposition 3), a sharp estimate from below of the area of a surface whose boundary is a fixed curve in the convex surface in $\mathbb{R}^3$ (which implies also well-known isoperimetric inequalities) with some non-sharp higher-dimensional version (Theorems 4, 4', 5), an inverse Krasnoselskii inequality for star-shaped subsets of a given domain (Theorem 6), a universal bound from *above* on the area of the boundary of a convex domain in terms of the volume and the injectivity radius, see Theorem 8 (the estimate from below was obtained by C. Croke, see [Cr2]). We also give an estimate of the first eigenvalue of the Dirichlet problem in the case of non-positive curvature and a sharp estimate of the first eigenvalue of the harmonic oscillator operator in a Hadamard manifold (Theorems 10, 11). We refer to [Re3] for the relation to the generating objects of symplectomorphisms of coadjoint orbits in real Lie algebras and to [Re4] for the construction of new metrics with integrable geodesic flow.

1.

Let X be a closed compact domain with the smooth boundary in some Riemannian manifold of M of dimension n. We will call X convex if every two points of X can be joined by a unique geodesic segment lying in X, and if this geodesic segment is length-minimizing and its interior points lie in $\mathrm{Int}X = X \sim \partial X$. It easily follows that X is diffeomorphic to the unit n-dimensional ball D_n. Let

CX be the set of all geodesics in M having nonempty intersection with X. It is clear that CX is a union of two sets which are in one-to-one correspondence with $X \times X \sim \Delta$ and $U(\partial X)$, respectively, where Δ is the diagonal and $U(\partial X)$ is the unit tangent bundle.

PROPOSITION 1: *CX carries a natural smooth structure and is diffeomorphic to the unit disk tangent bundle over ∂X.*

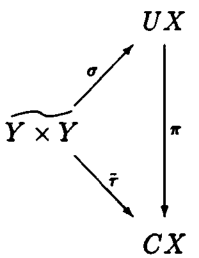
Indeed, it is sufficient to note that CX can be looked at as a result of a σ-process in every point of ∂C.

PROPOSITION 2: *There exists a natural symplectic structure on CX.*

Proof: Let $H : T^*X \to \mathbb{R}$ be the energy function determined by the Riemannian metric. We can consider $U(X)$ to be a level hypersurface of H, if we identify naturally the tangent and cotangent bundles (see [Be]). Then the space CX is identified with the result of the Weinstein–Marsden reduction (see [Ar]), and thus carries the symplectic structure, say ω. We will denote by $\pi : U(X) \to CX$ the natural projection and by α the Liouville 1-form on $U(X)$, so $d\alpha = \pi^*\omega$.

PROPOSITION 3: *Let $Y \subset X$ be a closed submanifold,* dim $Y = k$. *Consider a natural map $\tau : (Y \times Y) \sim \Delta \to CX$, where $\tau(y_1, y_2)$ is the geodesic segment with the ends y_1, y_2. Then $\int_{Y \times Y \sim \Delta} (\tau^*\omega)^k = C(k)\mathrm{Vol}_k Y$. The constant $C(k)$ is equal to $k!\mathrm{Vol}_k(D_k)$.*

Proof: As before, we can consider $Y \times Y \sim \Delta$ as an interior of a compact manifold $\widetilde{Y \times Y}$ with boundary $\partial(\widetilde{Y \times Y})$, which is naturally diffeomorphic to $U(Y)$. The natural map $\tilde{\tau} : \widetilde{Y \times Y} \to CX$ admits the following decomposition:

(1) 

where $\sigma(y_1, y_2)$ is the unit vector in $T_{y_1}X$, tangent to the geodesic segment $\tau(y_1, y_2)$. Hence $\tilde{\tau}^*\omega = \sigma^* d\alpha = d(\sigma^*\alpha)$, so $(\tilde{\tau}^*\omega)^k = d(\sigma^*(\alpha \wedge (d\alpha)^{k-1}))$. By the Stocks formula we have

$$\int_{\widetilde{Y \times Y}} (\tilde{\tau}^*\omega)^k = \int_{U(Y)} \sigma^*(\alpha \wedge (d\alpha)^{k-1}).$$

Note that there is the natural embedding, which we also denote σ, from $U(Y)$ to $U(X)$, and the Liouville form α behaves as a functorial object, i.e. $\sigma^*\alpha_X = \alpha_Y$, to the last integral is equal to $\int_{U(Y)} \alpha_Y \wedge (d\alpha_Y)^{k-1} = C(k)\mathrm{Vol}_k(Y)$ (see [Be]).

Remark: In the case $X = \mathbb{R}^n$ the statement of Proposition 3 is known as Pohl's formula (see [P]).

Applying our result to $Y = \partial X$ we obtain the following

PROPOSITION 4 (Santalo's formula):

$$\int_{CX} \omega^{n-1} = C(n-1)\mathrm{Vol}_{n-1}(\partial X). \tag{2}$$

THEOREM 1 ("Archimedes" inequality): *If $Z_1 \subseteq Z_2$ are two convex domains in X, then* $\mathrm{Vol}_{n-1}(\partial Z_1) \leq \mathrm{Vol}_{n-1}(\partial Z_2)$.

Proof: This is an immediate corollary of (2).

Consider an absolutely integrable function $f : U(X) \to \mathbb{C}$. By the Fubini theorem we have

$$\int_{U(X)} f \cdot \alpha \wedge (d\alpha)^{n-1} = \int_{CX} \left(\int_\gamma f(x,v)d\ell(x) \right) \omega^{n-1}, \tag{3}$$

where $\gamma \in CX$ is a geodesic segment, $x \in \gamma$, (x, v) means the unit tangent vector to γ at x, and $d\ell(x)$ is the length element on γ. Putting $f \equiv 1$, we obtain

PROPOSITION 5 (Crofton formula):

$$\int_{CX} \ell(\gamma)\omega^{n-1} = C(n)\mathrm{Vol}_n(X), \tag{4}$$

where $\ell(\gamma)$ is the length of the segment γ.

THEOREM 2 (C. Croke): *If $Z \subseteq X$ is a convex domain in X, then*

$$\mathrm{Vol}_n(Z) \leq \frac{C(n-1)}{C(n)} \cdot \mathrm{diam} Z \cdot \mathrm{Vol}_{n-1}(\partial Z). \tag{5}$$

Proof: It's enough to combine (4) and (2), and to note that $\ell(\gamma) \leq \mathrm{diam} Z$.

Let Q be a $(n-1)$-dimensional hypersurface in $\mathrm{Int} X$. Let $\tilde{U}(Q)$ denote the restriction of the bundle $U(X)$ to Q. Suppose $q \in Q$ and $(q, v) \in \tilde{U}(Q)$ and denote by $\pi(q, v)$ the geodesic in CX, tangent to (q, v). As in (1) it follows that $\pi^*\omega = d\alpha|_{\tilde{U}(Q)}$. Consider the fibered orthogonal projection $\eta : \tilde{U}(Q) \to$

TQ, then, similar to what was said before $\alpha|_{\tilde{U}(Q)} = \eta^*\alpha_Q$, where α_Q is the Liouville form in TQ itself. It follows that $(\pi^*\omega)^{n-1} = \eta^*(d\alpha_Q)^{n-1}$. Note that $(d\alpha_Q)^{n-1}$ is the canonical volume form in TQ (see [Be]), and admits the following decomposition: $(d\alpha_Q)^{n-1} = (n-1)!dw \wedge dq$, where dw is any $(n-1)$-form, whose restriction of T_qQ is equal to the euclidean volume form, and dq is the Riemannian volume form on Q. Let dv be any $(n-1)$-form on $\tilde{U}(Q)$, whose restriction on $UT_q(X)$ coincides with the spherical volume form, then it is clear that $\eta^*(d\alpha_Q)^{n-1} = (n-1)!\cos\theta(v)dv \wedge dq$, where $\theta(v)$ is the angle between v and the positive normal $n(q)$ to Q at q (we assume X and Q to be oriented). So we obtain the following

PROPOSITION 6 (Birkhoff formula): *On $\tilde{U}(Q)$ we have*

$$(\pi^*\omega)^{n-1} = (n-1)!\cos\theta(v)dv \wedge dq. \tag{6}$$

Of course, (6) implies (2). Moreover, it's easy now to obtain the following

PROPOSITION 7: *For $\gamma \in CX$ let $d_{\gamma,Q}$ be the number $\#(\gamma \cap Q)$. Then*

$$\mathrm{Vol}_{n-1}(Q) = \frac{1}{2C(n-1)} \int_{CX} d_{\gamma,Q}\omega^{n-1}. \tag{7}$$

We need some more formulas, expressing the volume form ω^{n-1} on CX via more usual forms and measures. Recall that, given two points y_1, y_2, we denote by $\tau(y_1,y_2)$ the geodesic, joining y_1 and y_2. Let W_i, $i = 1,\dots,n-1$, be Jacobi's fields along $\tau(y_1,y_2)$, such that $W_i(y_1) = 0$ and covariant derivatives $W_i'(y_1)$ form the orthonormal base in $T_{y_1}X$ along with the tangent vector to $\tau(y_1,y_2)$. We denote

$$J(y_1,y_2) = \mathrm{Vol}_{n-1}(W_1(y_2),\dots,W_{n-1}(y_2)), \tag{8}$$

where the right side means the volume of the spanned parallelotope. Recall ([Bu-Za]), that $J(y_1,y_2)$ is equal to the Jacobian of the exponential map $\exp : TX \to X$ in the appropriate point, and that $J(y_1,y_2) = J(y_2,y_1)$.

PROPOSITION 8: $(n-1)!dy_1 \wedge dy_2 = J(y_1,y_2)d\ell(y_1) \wedge d\ell(y_2) \wedge \tau^*\omega^{n-1}$ *on* $\mathrm{Int}X \times \mathrm{Int}X$, *where $d\ell(y_1) \wedge d\ell(y_2)$ is any 2-form, whose restriction on any fiber $\tau^{-1}(\gamma)$ coincides with the euclidean area form on the flat square.*

Proof: It follows immediately from the explicit expression for ω in terms of Jacobi's fields (see [Be]).

As a corollary we obtain

$$(n-1)!\mathrm{Vol}_n^2(X) = \int_{CX} \left(\int_{\tau(y_1,y_2)=\gamma} J(y_1,y_2)d\ell(y_1)d\ell(y_2) \right) \omega^{n-1}, \tag{9}$$

and we note that $\tau(y_1, y_2) = \gamma$ is equivalent to $y_1, y_2 \in \Gamma$. In the case of a flat $X \subseteq \mathbb{R}^n$, (9) takes the form (Crofton)

$$(n-1)!\mathrm{Vol}_n^2 X = \frac{1}{n(n+1)} \int_{CX} \ell^{n+1}(\gamma)\omega^{n-1}. \tag{10}$$

Now suppose y_1, y_2 lie on the fixed hypersurface Q. In what follows it is more convenient to denote $q_i = y_i$.

PROPOSITION 9:

$$dq_1 \wedge dq_2 = \frac{1}{(n-1)!} \frac{J(q_1,q_2)}{\cos\theta(q_1)\cdot\cos\theta(q_2)} \tau^*\omega^{n-1} \quad \text{on } Q \times Q,$$

where $\theta(q_1) = \theta(v_1)$, v_1 tangent to $\tau(q_1, q_2)$ at q_1, and the same for q_2.

Proof: It easily follows from Proposition 8.

As a corollary we obtain

$$\mathrm{Vol}_{n-1}^2(\partial X) = \frac{1}{(n-1)!} \int_{CX} \frac{J(q_1,q_2)}{\cos\theta(q_1)\cos\theta(q_2)} \omega^{n-1}, \tag{11}$$

where q_1 and q_2 are the ends of $\gamma \in CX$.

In the flat case $X \subset \mathbb{R}^n$ we obtain

$$\mathrm{Vol}_{n-1}^2(\partial X) = \frac{1}{(n-1)!} \int_{CX} \frac{\ell^{n-1}(\gamma)}{\cos\theta(q_1)\cos\theta(q_2)} \omega^{n-1}. \tag{12}$$

2.

We are not in a position to make use of the formulas above.

THEOREM 3: *Suppose X to have non-positive curvature. Then for some constant $B(n)$ the classical isoperimetrical inequality holds:*

$$\mathrm{Vol}_n(X) \le B(n)(\mathrm{Vol}_{n-1}(\partial X))^{n/(n-1)}. \tag{13}$$

When $n = 2, 4$, then the same inequality holds with the sharp constant $n^{n/(1-n)}(\mathrm{Vol}_n(D_n))^{1/(1-n)}$.

Proof: The proof is the consequence of (2), (4), (10), the Hölder inequality and the Rauch inequality $J(y_1, y_2) \ge \rho^{n-1}(y_1, y_2)$, where $\rho(\cdot,\cdot)$ denotes the distance function related to the metric. We omit the details because, as it was said, similar arguments can be found in [Cr1]. See [Re1] also.

A definite gap in orders of $l(\gamma)$ in the formulas (2), (4), (10) is felt when attempting to get the sharp constant in (13) for $n > 4$. It is not clear whether this gap can be filled with extra formulas of this type.

The following theorem gives the sharp estimate from below of the volume of a bubble, spanned on a given curve.

THEOREM 4: *Let β be a simple closed curve in the standard sphere $S^2 \subset \mathbb{R}^3$. Let F_1, F_2 be the areas of the two components Y_1, Y_2 of $S^2 \sim \beta$. Let Q be any smooth surface in $\mathbb{R}^3$ with $\partial Q = \beta$. Then*

$$\mathrm{Vol}_2(Q) \geq F_1 - \frac{F_1^2}{4\pi} = \frac{F_1 F_2}{4\pi}. \tag{14}$$

Remark: Take the minimal bubble Q. Then by the isoperimetrical inequality for the hyperbolic surfaces (see Theorem 3 above or [Bu-Za]),

$$\mathrm{Vol}_2(Q) \leq \frac{1}{4\pi}\mathrm{Vol}_1^2(\beta).$$

Together with (14) this gives the spherical isoperimetrical inequality.

Proof: Let $R \subset C\mathbb{R}^3$ be the set of all straight lines, linked with β. Then by (7),

$$\mathrm{Vol}_2(Q) \geq \frac{1}{4\pi}\int_R \omega^2,$$

because for every $\gamma \in R$, $\gamma \cap Q \neq \emptyset$. From the other side it is clear that $\gamma \in R$ if and only if $d_{\gamma,Y_1} = d_{\gamma,Y_2} = 1$. Applying (7) to Y_1 we see

$$F_1 = \frac{1}{4\pi}\left(\int_{d_{\gamma,Y_1}=1} \omega^2 + 2\int_{d_{\gamma,Y_1}=2} \omega^2\right),$$

so

$$\mathrm{Vol}_2(Q) \geq F_1 - \frac{1}{2\pi}\int_{d_{\gamma,Y_1}=2} \omega^2.$$

Let us use Proposition 9 to compute the last integral:

$$\int_{d_{\gamma,Y_1}=2} \omega^2 = 2\int_{Y_1\times Y_1} \frac{\cos\theta(y_1)\cos\theta(y_2)}{\rho^2(y_1,y_2)} dy_1 \wedge dy_2,$$

where $\rho(y_1, y_2)$ is the distance in $\mathbb{R}^3$. Elementary geometry shows that

$$\cos\theta(y_1) = \cos\theta(y_2) = \frac{\rho(y_1, y_2)}{2},$$

hence $\int_{d_{\gamma,Y_1}=2} \omega^2 = \frac{1}{2}F_1^2$. ∎

If we replace S^2 by any convex surface M, we will obtain similar results involving the curvature of M.

THEOREM 4': *(a) Suppose all the main curvatures $\lambda_i(M)$, $i = 1, 2$ are bounded by 1 from below everywhere in M, then, using the same notations*

$$\mathrm{Vol}_2(Q) \geq \frac{F_1 F_2}{4\pi}.$$

(b) Suppose $\lambda_i(M)$ are bounded by 1 from above, then

$$\mathrm{Vol}_2(Q) \geq F_1 - \frac{F_1^2}{4\pi}.$$

Remark: Again applying this to the minimal bubble, we will obtain in case (a) Paul Lévy's isoperimetrical inequality in dimension 2, and in case (b) the isoperimetrical inequality

$$\frac{\mathrm{Vol}_1^2(\beta)}{4\pi} \geq F_1 - \frac{F_1^2}{4\pi}$$

(see [Bu-Za]).

Proof: We begin with (b) and act as in the previous theorem. To obtain the estimate of

$$J = \int_{Y_1 \times Y_1} \frac{\cos\theta(y_1)\cos\theta(y_2)}{\rho^2(y_1, y_2)} dy_1 \wedge dy_2$$

we note that by the Blaschke theorem (see [B]), $\rho(y_1, y_2) \geq 2\cos\theta(y_i)$, $i = 1, 2$. Thus we have $J \leq \frac{1}{4}F_1^2$, which proves our statement.

In the case (a) we note that $\gamma \in R$ if and only if $d_{\gamma,Y_1} = d_{\gamma,Y_2} = 1$, hence by Proposition 9, taking into account the possibilities $y_1 \in Y_1$, $y_2 \in Y_2$ and $y_1 \in Y_2$, $y_2 \in Y_1$, we have

$$\int_R \omega^2 = 4 \int_{Y_1 \times Y_1} \frac{\cos\theta(y_1)\cos\theta(y_2)}{\rho^2(y_1, y_2)} dy_1 \wedge dy_2.$$

Again using the Blaschke theorem, now in the form $\rho(y_1, y_2) \leq 2\cos\theta(y_i)$, we obtain $\int_R \omega^2 \geq F_1 F_2$. ∎

In higher dimensions we obtained the following estimate, which is not sharp.

THEOREM 5: *Let β be the smooth embedded $(n-2)$-dimensional sphere $S^{n-1} \subset \mathbb{R}^n$, and let F_1, F_2, Q be as in the previous theorems, then*

$$\mathrm{Vol}_{n-1}(Q) \geq F_1 - G(n)F_1^{\frac{n-1}{n-2}},$$

where

$$G(n) = \frac{1}{4\mathrm{Vol}_{n-1}(D_{n-1})}(n\mathrm{Vol}_n(D_n))^{\frac{n-3}{n-2}}.$$

Proof: As before, we reduce the statement of the theorem to the estimating of the integral

$$I = \int_{Y_1 \times Y_1} \frac{\cos\theta(y_1)\cos\theta(y_2)}{\rho^{n-1}(y_1, y_2)} dy_1 \wedge dy_2.$$

Fix y_1 and denote $\rho(y_1, y_2) = \mu(y_2)$, then $\cos\theta(y_1) = \cos\theta(y_2) = \frac{1}{2}\mu(y_2)$. As $dy_2 \cos\theta(y_2) = \rho^{n-1}(y_1, y_2)dv$ (here dv, as before, is the spherical volume form in $UT_{y_1}\mathbb{R}^n$), we have

$$F_1 = \int_{Y_1} dy_2 = 2\int_{\sum} \mu^{n-2}(v)dv$$

and

$$K = \int_{Y_1} \frac{\cos\theta(y_1)\cos\theta(y_2)}{\rho^{n-1}(y_1, y_2)} dy_2 = \frac{1}{2}\int_{\sum} \mu(v)dv.$$

Here, $\sum \subset UT_{y_1}\mathbb{R}^n$ denotes the set of all v corresponding to $y_2 \in Y_1$ by $\pi(y_1, v) = \tau(y_1, y_2)$. By the Hölder inequality we have

$$K \leq \frac{1}{2}\left(\frac{F_1}{2}\right)^{\frac{1}{n-2}} \times \left(\frac{\mathrm{Vol}_{n-1}(S^{n-1})}{2}\right)^{\frac{n-3}{n-2}},$$

hence

$$I \leq \tfrac{1}{4}(\mathrm{Vol}_{n-1}(S^{n-1}))^{\frac{n-3}{n-2}} F_1^{\frac{n-1}{n-2}},$$

and then as in Theorem 4.

Theorems 4, 4', 5 could be viewed as the answer to R. Osserman's question [Os] on the interpretation of the Banchoff–Pohl inequality [B-P].

Let Y be any, not necessarily convex, subdomain in X. We are going to apply the analogue of the Crofton formula (4) in this situation. For this we apply (3) to the characteristic function $\chi(Y)$ of Y and obtain $C(n)\mathrm{Vol}_n(Y) = \int_{CX} \ell(\gamma \cap Y)\omega^{n-1}$. Consider again the restriction $\tilde{U}(\partial Y)$ of the unit tangent bundle $U(X)$ on $Q = \partial Y$ with the canonical volume form $\cos\theta(v)dv \wedge dq$ and let $\pi : \tilde{U}(\partial Y) \to CX$ be as before. For any $(q,v) \in \tilde{U}(\partial Y)$ let $\ell(q,v)$ mean the length of the segment of the geodesic $\pi(q,v)$, being inside Y till the first after q intersection point with ∂Y. Then (see also [Cr2], [Ber1])

$$\frac{1}{(n-1)!}C(n)\mathrm{Vol}_n(Y) = \int_{\tilde{U}(\partial Y)} \ell(q,v)\cos\theta(v)dv \wedge dq. \tag{15}$$

Indeed, this is the direct consequence of the previous formula and the Kronrod formula (see [Bu-Za]), applied to the map $\pi : \tilde{U}(\partial Y) \to CX$. Let $\mathrm{St}(q)$ be the set of all $y \in Y$, such that the shortest geodesic segment $\tau(q,y)$ lies in Y. It is natural to call it the star-shaped neighborhood of q.

THERQREM 6: *Suppose the curvature of X is nonpositive. Then there exists $q \in \partial Y$, such that*

$$\mathrm{Vol}_n(\mathrm{St}(q)) \geq 2\mathrm{Vol}_{n-1}S^{n-1}\left(\frac{\mathrm{Vol}_n(Y)}{\mathrm{Vol}_{n-1}(\partial Y)}\right)^n. \tag{16}$$

Remark: Let us look at the simplest case $X = \mathbb{R}^n$. Suppose Y is not star-shaped, i.e. $Y \neq \mathrm{St}(q)$ for any q. Then by the Krasnoselskii theorem (see [Ber3]) there exists $q \in \partial Y$, such that

$$\mathrm{Vol}_n(\mathrm{St}(q)) \leq \frac{n}{n+1}\mathrm{Vol}_n(Y).$$

Our theorem gives an estimate from below on the volume of $\mathrm{St}(q)$.

Proof: From (15) we see that there exists such $q \in \partial Y$, that

$$U_q(X) = UT_q(X)\int_{U_q(X)} \ell(q,v)\cos\theta(v)dv \geq \frac{1}{(n-1)!}C(n)\frac{\mathrm{Vol}_n(Y)}{\mathrm{Vol}_{n-1}(\partial Y)}.$$

By the Hölder inequality and $\cos\theta(v) \leq 1$ we estimate the left side by

$$\left(\int_{U_q(X)} \ell^n(q,v)dv\right)^{1/n} \times \left(\frac{\mathrm{Vol}_{n-1}(S^{n-1})}{2}\right)^{\frac{n-1}{n}},$$

because actually the integration is over a half-sphere only. Consider the exponential map $\exp_q : T_qX \to X$ and let $\mathrm{st}(q) = \exp_q^{-1}(\mathrm{St}(q))$. Then evidently

$$\mathrm{Vol}_n(\mathrm{st}(q)) = \int_{U_q(X)} \ell^n(q,v)dv$$

and by the curvature condition $\mathrm{Vol}_n(\mathrm{st}(g)) \leq \mathrm{Vol}(\mathrm{St}(q))$. So

$$(\mathrm{Vol}_n(\mathrm{St}(q)))^{1/n} \cdot \left(\frac{\mathrm{Vol}_{n-1}(S^{n-1})}{2}\right)^{\frac{n-1}{n}} \geq \mathrm{Vol}_{n-1}(S^{n-1})\frac{\mathrm{Vol}_n(Y)}{\mathrm{Vol}_{n-1}(\partial Y)},$$

because $C(n) = n!\mathrm{Vol}_n(D_n) = (n-1)!\mathrm{Vol}_{n-1}(S^{n-1})$. ∎

Formula (15) still holds when Y is a domain in a *closed* oriented manifold X of negative curvature. Of course one cannot introduce any smooth structure on the set CX of all geodesics in this situation. But locally it can be done (take a convex domain X' and consider CX'). Let us define the map $\pi : \tilde{U}(\partial Y) \to CX$ as before. If $x \in \gamma = \pi(q, v)$ then by the curvature condition, all geodesics close to γ near x can be represented as $\pi(q_1, v_1)$ for (q_1, v_1) close to (q, v), if only γ is transversal to ∂Y at q. Moreover, by the conservation law for the symplectic product of Jacobi's field's (see [Be]) we see that $\pi^*\omega^{n-1} = (n-1)! \cos\theta(v) dv \wedge dq$. Consider the bundle $I_{q,v} \to E \xrightarrow{\epsilon} \tilde{U}_+(\partial Y)$, where $I_{q,v} = [0, \ell(q, v)]$ and the fiber over (q, v) is the geodesic segment in $\pi(q, v)$ between q and the next intersection point with ∂Y. By $\tilde{U}_+(\partial Y)$ we mean those (q, v) for which $|\theta(v)| < \frac{\pi}{2}$, so $\ell(q, v) > 0$. Denote by $\hat{\pi} : E \to U(Y)$ the map

$$\hat{\pi}(t, q, v) = \left(\frac{d}{ds}\pi(q, v)\right)_{s=t}.$$

We can pull the Liouville form α back to E, say $\beta = \hat{\pi}^*\alpha$, then $d\beta = \epsilon^*(\pi^*\omega)$ because $d\alpha = \pi^*\omega$ and $\pi \circ \hat{\pi} = \pi \circ \epsilon$. For any function $\hat{f}$ on E we will have

$$\int_E \hat{f}\beta \wedge (d\beta)^{n-1} = \int_{\tilde{U}_+(\partial Y)} \left(\int_{I_{q,v}} \hat{f}\beta\right) \pi^*\omega^{n-1}.$$

For a function f on $U(Y)$ let $\hat{f} = f \circ \hat{\pi}$. Note that the image $\hat{\pi}(E)$ has the full measure in $U(Y)$ by the Anosov–Sinai theory (see [An]). This proves (15).

As a corollary, we obtain the following statement.

THEOREM 7: *Let X be a closed hyperbolic manifold and let Y be a compact domain in X with a smooth boundary ∂Y. Then there exist such (q_1, v_1), $(q_2, v_2) \in \tilde{U}(\partial Y)$, that*

$$\ell(q_1, v_1) \geq \frac{C(n)}{C(n-1)} \frac{\mathrm{Vol}_n(Y)}{\mathrm{Vol}_{n-1}(\partial Y)} \geq \ell(q_2, v_2). \tag{17}$$

If we look at an arbitrary Riemannian manifold, we can only say that $\hat{\pi} : E \to UY$ is an immersion, so instead of (15) we will have only an inequality

$$\frac{1}{(n-1)!}C(n)\mathrm{Vol}_n(Y) \geq \int_{\bar{U}_+(\partial Y)} \ell(q,v)\cos\theta(v)dv \wedge dq$$

(see [Cr2]) and therefore the right side of (17) still holds for some (q_2, v_2).

THEOREM 8: *Let X be a closed oriented Riemannian manifold with the injectivity radius $r_i(X)$ and let Y be a convex domain in X such that* $\operatorname{diam} Y < r_i(X)$. *Then*

$$\frac{C(n)}{C(n-1)}\frac{\mathrm{Vol}_n(Y)}{r_i(X)} \leq \mathrm{Vol}_{n-1}(\partial Y) \leq \frac{C(n)}{C(n-1)}\frac{\mathrm{Vol}_n(X)}{r_i(X)}. \tag{18}$$

Proof: The left side is actually proved in Theorem 2. Now applying the right side of (17) to $X \sim Y$, consider a geodesic segment γ in $X \sim Y$ with the ends $x, y \in \partial Y$ of the length no more than

$$\mu = \frac{C(n)}{C(n-1)}\frac{\mathrm{Vol}_n(X \sim Y)}{\mathrm{Vol}_{n-1}(\partial Y)}.$$

By convexity of Y, there exists a geodesic segment γ_1 in Y with the same ends and its length is no more than $\operatorname{diam} Y < r_i(X)$. By definition of $r_i(X)$ if follows that $\mu \geq r_i(X)$. ∎

3.

In this section we work with eigenvalues of the Laplace and Schrödinger operators on Riemannian manifolds. If $X \subseteq M$ is a compact domain with a smooth boundary ∂X, then by $\lambda_1(X)$ we denote the first eigenvalue of the Dirichlet problem $-\Delta f = \lambda f$, $f|_{\partial X} = 0$. We need to reproduce first the result of C. Croke [Cr2].

THEOREM 9: *Suppose X is convex, then*

$$\lambda_1(X) \geq \pi^2 n \frac{1}{\mathrm{diam}^2(X)}. \tag{19}$$

Proof: Let $g \in C_0^\infty(X)$, i.e. g is smooth and its support lies in $\mathrm{Int}X$,and let $f \in C_0^\infty(U(X))$ is defined as $f(x,v) = |dg_x(v)|^2$. Applying (3) we have

$$\int_{U(X)} f(x,v)\alpha \wedge (d\alpha)^{n-1} = \int_{CX}\left(\int_\gamma \left(\frac{dg}{d\sigma}\right)^2 d\sigma\right)\omega^{n-1},$$

where σ is length parameter on γ. By the Poincaré inequality,

$$\int_\gamma \left(\frac{dg}{d\sigma}\right)^2 d\sigma \geq \frac{\pi^2}{\ell^2(\gamma)} \int_\gamma g^2 d\sigma = \pi^2 \int_\gamma \frac{1}{\ell^2(x,v)} g^2(x) d\sigma,$$

where (x, v) is tangent to γ (so $\ell(x, v) = \ell(\gamma)$). Denote

$$h(x,v) = \frac{g(x)}{\ell(x,v)},$$

then the last integral can be rewritten as $\pi^2 \int_\gamma h^2(x, v) d\sigma$, so, again applying (3) we have

$$\int_{CX} \left(\int_\gamma \left(\frac{dg}{d\sigma}\right)^2 d\sigma\right) \omega^{n-1} \geq \pi^2 \int_{U(X)} h^2(x,v) \alpha \wedge (d\alpha)^{n-1}.$$

From the other side, $\alpha \wedge (d\alpha)^{n-1} = (n-1)! dv \wedge dx$ (see [Be]), so

$$\int_X dx \left(\int_{U_x(X)} |dg_x(v)|^2 dv\right) \geq \pi^2 \int g^2(x) dx \left(\int_{U_x(X)} \frac{dv}{\ell^2(x,v)}\right).$$

Next, for any linear form ψ,

$$\int_{S^{n-1}} |\psi(v)|^2 dv = \mathrm{Vol}_n(D_n) \cdot |\psi|^2,$$

so

$$\mathrm{Vol}_n(D_n) \int_X |dg_x|^2 dx \geq \pi^2 \int_X g^2(x) dx \left(\int_{U_x(X)} \frac{dv}{\ell^2(x,v)}\right). \tag{20}$$

The integral in the brackets is not less than $\frac{1}{\mathrm{diam}^2(X)} \mathrm{Vol}_{n-1}(S^{n-1})$, so, finally,

$$\int_X |dg_x|^2 \geq \frac{\pi^2}{\mathrm{diam}^2(X)} \frac{\mathrm{Vol}_{n-1}(S^{n-1})}{\mathrm{Vol}_n(D_n)} \int_X g^2(x) dx,$$

which is equivalent to (19) by the minimax principle and

$$\frac{\mathrm{Vol}_{n-1}(S^{n-1})}{\mathrm{Vol}_n(D_n)} = n.$$

THEOREM 10: *Suppose $X \subseteq Y$ where Y is convex and has non-positive curvature, then*

$$\lambda_1(X) \geq \pi^2 n \frac{(\mathrm{Vol}_n(D_n))^{2/n}}{(\mathrm{Vol}_n(X))^{2/n}}. \tag{21}$$

Proof: Despite the fact that X may be non-convex, we can do as in the previous theorem following the construction before Theorem 7. Now we estimate the integral in the brackets in (20) as

$$\begin{aligned}\int_{U_x(X)} \frac{dv}{\ell^2(x,v)} &\geq \frac{(\mathrm{Vol}_{n-1}(S_{n-1}))^2}{\int_{U_x(X)} \ell^2(x,v)dv} \\ &\geq \frac{\mathrm{Vol}^2_{n-1}(S^{n-1})}{\left(\int_{U_x(X)} \ell^n(x,v)dv\right)^{2/n} (\mathrm{Vol}_{n-1}(S^{n-1}))^{\frac{n-2}{n}}} \\ &= \frac{\mathrm{Vol}_{n-1}^{\frac{n+2}{n}}(S^{n-1})}{\left(\int_{U_x(X)} \ell^n(x,v)dv\right)^{2/n}},\end{aligned}$$

by the Cauchy inequality and Hölder inequality. Suppose $\ell_+(v)$ and $\ell_-(v)$ are the lengths of the two components of $\pi(x,v) \sim \{x\}$, then

$$\begin{aligned}&\left(\left(\int_{U_x(X)} \ell^n(x,v)dv\right)^{1/n}\right)^2 \\ &\leq \frac{1}{2}\left[\left(\int_{U_x(X)} \ell_+^n(x,v)dv\right)^{2/n} + \left(\int_{U_x(X)} \ell_-^n(x,v)\right)^{2/n}\right]\end{aligned}$$

by the Minkowski inequality. Each of these integrals is no more than $n\mathrm{Vol}_n(X)$ by the curvature condition, so we obtain the following final estimate:

$$\int_{U_x(X)} \frac{dv}{\ell^2(x,v)} \geq \frac{\mathrm{Vol}_{n-1}^{\frac{n+2}{n}}(S^{n-1})}{n^{2/n}(\mathrm{Vol}_n(X))^{2/n}},$$

which together with $\mathrm{Vol}_{n-1}(S^{n-1}) = n\mathrm{Vol}_n(D_n)$ completes the proof.

Remark: If we had the sharp constant in the isoperimetrical inequality (13), we would be able to apply the Faber–Krahn–Bérard–Gallot [Bd] symmetrisation argument to get the sharp constant in (21). For the time being, however, the constant in (21) seems to be the best possible at least for $n > 4$. I am thankful to the referee for pointing this out to me.

Our last result deals with the Schrödinger operators. Suppose X to be a Hadamard (= of nonpositive curvature, simply connected) manifold and fix a "pole", $p \in X$. We introduce the *harmonic oscillator operator* H in $L^2(X, dx)$ by the formula $Hf = -\Delta f + \rho^2(p, \cdot)f$. When $X = \mathbb{R}^n$, its spectrum is well-known to be $(n, n+2, n+4, \ldots)$ and the eigenfunctions are just Hermite functions (see [R-S] and also [W] for beautiful topological applications). Our method leads to a *sharp* estimate of the spectrum in the hyperbolic case.

THEOREM 11: *With all the assumptions above,* $\lambda_1(H) \geq n$.

Proof: We begin as in Theorem 9 with an identity

$$I = \int_{U(X)} (\rho^2(p,x)g^2(x) + (dg_x(v))^2)\alpha \wedge (d\alpha)^{n-1}$$
$$= \int_{CX} \left(\int_\gamma \left(\rho^2(p,x)g^2(x) + \left(\frac{dg}{d\sigma}\right)^2 \right) d\sigma \right) \omega^{n-1}.$$

Let $x_0 \in \gamma$ be the closest to p point in γ, then by the hyperbolicity,

$$\rho^2(p,x) \geq \rho^2(p,x_0) + \rho^2(x_0,x),$$

so

$$I \geq \int_{CX} \left(\int_\gamma \left((\rho^2(p,x_0) + \rho^2(x_0,x))g^2(x) + \left(\frac{dg}{d\sigma}\right)^2 \right) d\sigma \right) \omega^{n-1}.$$

From the theory of the one-dimensional harmonic oscillator we know that

$$\lambda_1 \left(-\frac{d^2}{dx^2} + x^2 \right) \geq 1,$$

so

$$I \geq \int_{CX} \left(\int_\gamma (\rho^2(p,x_0) + 1)g^2(x)d\sigma \right) \omega^{n-1}. \tag{22}$$

We consider $\rho(p, x_0)$ to be a function of the unit vector (x, v) tangent to γ and denote it $\rho(x, v)$. Of course, $\rho(x, v)$ is the distance between ρ and the geodesic $\pi(x, v)$. Applying (22), we have

$$I \geq \int_{U(X)} (\rho^2(x,v) + 1)g^2(x)\alpha \wedge (d\alpha)^{n-1}$$
$$= n!\mathrm{Vol}_n(D_n) \int_X g^2(x)dx + (n-1)! \int_X g^2(x)dx \left(\int_{U_x(X)} \rho^2(x,v)dv \right).$$

By the lemma below,

$$\int_{U_x(X)} \rho^2(x,v)dv \geq (n-1)\mathrm{Vol}_n(D_n)\rho^2(p,x),$$

so

$$I \geq n!\mathrm{Vol}_n(D_n)\int_X g^2(x)dx + (n-1)(n-1)!\int_X \rho^2(p,x)g^2(x)dx.$$

But

$$I = n!\mathrm{Vol}_n(D_n)\int_X \rho^2(p,x)g^2(x)dx + (n-1)!\mathrm{Vol}_n(D_n)\int_X |dg_x|^2dx,$$

hence

$$\int_X |dg_x|^2dx + \int_X \rho^2(p,x)g^2(x)dx \geq n\int_X g^2(x)dx,$$

which together with the minimax principle proves the theorem.

LEMMA: *Let ABC be a rectangular ($\angle BAC = \pi/2$) geodesic triangle in a Hadamard manifold M. Then $|AB| \geq |BC|\sin\angle ACB$.*

Proof (G. Perelman): Let us cut the side BC into $N > 1$ equal segments, say $Q_0Q_1, \ldots, Q_{N-1}Q_N$, where $Q_0 = B$, $Q_N = C$. Let $P_i \in AC$ be such that the geodesics $Q_iP_i \perp AC$. As the sum of angles of any triangle $\leq \pi$, we have $\angle P_iQ_iC \geq \frac{\pi}{2} - \angle C$. Let $R_i \in Q_iP_i$ be such that the geodesics $Q_{i+1}R_i \perp Q_iP_i$. Then

$$\angle Q_iQ_{i+1}R_i = \frac{\pi}{2} - \angle P_iQ_iC + O\left(\frac{1}{N^2}\right),$$

so

$$|Q_iR_i| \geq |Q_iQ_{i+1}|\sin\angle C + O\left(\frac{1}{N^2}\right).$$

It is clear that

$$|Q_iP_i| - |Q_{i+1}P_{i+1}| = |Q_iR_i| + O\left(\frac{1}{N^2}\right),$$

hence

$$|Q_0P_0| \geq \sum |Q_iQ_{i+1}|\sin\angle C + O\left(\frac{1}{N}\right).$$

We will obtain the inequality $|AB| \geq |BC|\sin\angle C$ when $N \to \infty$.

This lemma implies that in the proof of the theorem $\rho(x,v) \geq \rho(p,x)\sin\theta(v)$, where $\theta(v)$ is the angle between v and the vector tangent to the geodesic px at x. The inequality that we need follows from the formula

$$\int_{S^{n-1}} \sin^2\theta(v) = (n-1)\mathrm{Vol}_n(D_n).$$

References

[Ar] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, Berlin, 1978.

[Be] A. Besse, *Manifolds, All of Whose Geodesics are Closed*, Springer, Berlin, 1978.

[Ber1] M. Berger, *Lectures on Geodesics in Riemannian Geometry*, Tata Institute, Bombay, 1965.

[Ber2] M. Berger, *Some relations between volume, injectivity radius and convexity radius in Riemannian manifolds*, in *Differential Geometry and Relativity*, D. Reidel, Dordrecht, 1976, pp. 33–42.

[Ber3] M. Berger, *Geometry I*, Springer, Berlin, 1987.

[Bd] P. H. Bérard, *Spectral Geometry: Direct and Inverse Problems*, with Appendix by G. Besson, Lecture Notes in Math., Vol. 1207, Springer-Verlag, Berlin, 1986.

[B] W. Blaschke, *Kreis und Kugel*, de Gruyter, Berlin, 1956.

[Bu-Za] Yu. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Springer, Berlin, 1988.

[B-P] T. F. Banchoff and W. F. Pohl, *A generalization of the isoperimetrical inequality*, J. Differ. Geom. **6** (1971), 175–192.

[Cr1] C. Croke, *A sharp four dimensional isoperimetric inequality*, Comment. Math. Helv. **59** (1984), 187–192.

[Cr2] C. Croke, *Some isoperimetrical inequalities and eigenvalue estimates*, Ann. Sci. Éc. Norm. Sup. 4ᵉ série **13** (1980), 419–435.

[G-S] V. Guillemin and S. Sternberg, *Geometric asymptotics*, Math. Surveys, No. 14, AMS, 1977.

[O] R. Osserman, *The isperimetric inequality*, Bull. Am. Math. Soc. **87** (1978), 1182–1238.

[P] W. F. Pohl, *On some integral formulas for space curves and their generalizations*, Am. J. Math. **90** (1968), 1321.

[Re1] A. Reznikov, *Affine symplectic geometry*, preprint, Dŭkstas, Lithuania, 1986.

[Re2] A. Reznikov, *On the spectra of the manifolds, all of whose geodesics are closed*, Ukrainian Math. J. **38** (1986), 385–388 (Russian).

[Re3] A. Reznikov, *Non-commutative Gauss map*, Compositio Math. **83** (1992), 53–68.

[Re4] A. Reznikov, *Twisted ellipsoids and other surfaces with integrable geodesic flow*, preprint, 1992.

[R-S] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II*, Academic Press, New York, 1975.

[San] L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, Reading, Mass., 1976.

[W] E. Witten, *Supersymmetry and Morse theory*, J. Differ. Geom. **17** (1982), 661–692.